Chem 542 Problem Set 7

1. Can you show that:

$$i\hbar \int \psi^*(x') \cdot \psi(x') \cdot \partial x' = i\hbar$$

using Dirac notation? You can assume that $\langle\psi|\psi\rangle = 1$. Hint: You can reverse engineer this problem quite easily.

Answer: You should recognize you are observing a resolution of the identity:

$$\int \psi^*(x') \cdot \psi(x') \cdot \partial x' = \int \langle\psi|x'\rangle\langle x'|\psi\rangle \cdot \partial x'$$

Now you can remove $\int |x'\rangle\langle x'| \cdot \partial x' = 1$ leaving $\langle\psi|\psi\rangle = 1$. Hence, the constants remain.

2. Can you show that $\langle\psi|[\hat{x},\hat{p}]|\psi\rangle = i\hbar$? Please use Dirac notation to solve this problem, which requires double resolutions of the identity (in x' and x''). I suggest you use the resolutions to "bracket" the $\hat{p}$ operator with $x'$and x'' so that you can use the following:

$$\langle x'|p|x''\rangle = \left\{-i\hbar\frac{\partial}{\partial x'}\right\}\delta(x' - x'')$$

Hint: We assume the state ψ is normalized, i.e.: $\langle\psi|\psi\rangle = 1$.

Answer: $\langle\psi|[\hat{x},\hat{p}]|\psi\rangle = \langle\psi|\hat{x}\cdot\hat{p}|\psi\rangle - \langle\psi|\hat{p}\cdot\hat{x}|\psi\rangle$. Next we doubly resolve identities:

$$\langle\psi|[\hat{x},\hat{p}]|\psi\rangle = \int\int \langle\psi|\hat{x}|x'\rangle\langle x'|\hat{p}|x''\rangle\langle x''|\psi\rangle \cdot \partial x'\,\partial x'' - \int\int \langle\psi|x'\rangle\langle x'|\hat{p}|x''\rangle\langle x''|\hat{x}|\psi\rangle \cdot \partial x'\,\partial x''$$

$$= \int\int x'\langle\psi|x'\rangle\langle x'|\hat{p}|x''\rangle\langle x''|\psi\rangle \cdot \partial x'\,\partial x'' - \int\int \langle\psi|x'\rangle\langle x'|\hat{p}|x''\rangle x''\langle x''|\psi\rangle \cdot \partial x'\,\partial x''$$

Let's do the first one first:

$$\int\int x' \cdot \psi^*(x') \cdot \left\{-i\hbar\frac{\partial}{\partial x'}\right\}\delta(x' - x'') \cdot \psi(x'') \cdot \partial x'\,\partial x'' = -i\hbar\int x' \cdot \psi^*(x') \cdot \frac{\partial\psi(x')}{\partial x'} \cdot \partial x'$$

Likewise the other one is:

$$\int\int \langle\psi|x'\rangle\langle x'|\hat{p}|x''\rangle x''\langle x''|\psi\rangle \cdot \partial x'\,\partial x'' =$$

$$\int\int \psi^*(x')\cdot\left\{-i\hbar\frac{\partial}{\partial x'}\right\}\delta(x'-x'')\cdot x''\cdot\psi(x'')\cdot\partial x'\,\partial x''$$
$$= -i\hbar\int \psi^*(x')\cdot\frac{\partial}{\partial x'}\cdot\{x'\cdot\psi(x')\}\cdot\partial x'$$

You have to use the product rule on the above:

$$-i\hbar\int \psi^*(x')\cdot\frac{\partial}{\partial x'}\cdot\{x'\cdot\psi(x')\}\cdot\partial x'$$
$$= -i\hbar\int \psi^*(x')\cdot\psi(x')\cdot\partial x' - i\hbar\int x'\cdot\psi^*(x')\cdot\frac{\partial\psi(x')}{\partial x'}\cdot\partial x'$$

Now when you put it all together:

$$-i\hbar\int x'\cdot\psi^*(x')\cdot\frac{\partial\psi(x')}{\partial x'}\cdot\partial x'$$
$$-\left\{-i\hbar\int \psi^*(x')\cdot\psi(x')\cdot\partial x' - i\hbar\int x'\cdot\psi^*(x')\cdot\frac{\partial\psi(x')}{\partial x'}\cdot\partial x'\right\}$$
$$= i\hbar\int \psi^*(x')\cdot\psi(x')\cdot\partial x' = i\hbar$$

due to the fact that: $\int \psi^*(x')\cdot\psi(x')\cdot\partial x' = 1$.

3. Suppose $\hat{A}$ and $\hat{B}$ are two Hermitian operators. **a)** If:

$$\Delta\hat{A}|\psi\rangle = \lambda\cdot\Delta\hat{B}|\psi\rangle$$

where λ is purely imaginary, can you show that the uncertainty principle yields the lowest possible result:

$$\langle\psi|\left(\Delta\hat{A}\right)^2|\psi\rangle\langle\psi|\left(\Delta\hat{B}\right)^2|\psi\rangle = \frac{1}{4}\left|\langle\psi|\left[\hat{B},\hat{A}\right]|\psi\rangle\right|^2$$

Hint: First show that $\langle\psi|\left(\Delta\hat{A}\right)^2|\psi\rangle\langle\psi|\left(\Delta\hat{B}\right)^2|\psi\rangle = \left|\langle\psi\middle|\Delta\hat{B}\Delta A\middle|\psi\rangle\right|^2$. Next:

$$\left\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\right\rangle = \frac{1}{2}\left\langle\psi|\left[\Delta\hat{B},\Delta\hat{A}\right]|\psi\right\rangle + \frac{1}{2}\left\langle\psi|\left\{\Delta\hat{B},\Delta\hat{A}\right\}|\psi\right\rangle$$

where $\frac{\lambda^*}{\lambda} = -1$ is used to show that the second term (the anticommutator) is 0. Also recall from class that $\left\langle\psi|\left[\Delta\hat{B},\Delta\hat{A}\right]|\psi\right\rangle = \left\langle\psi|\left[\hat{B},\hat{A}\right]|\psi\right\rangle$, and based on the information given in the problem:

$$\Delta\hat{B}|\psi\rangle = \frac{1}{\lambda}\Delta\hat{A}|\psi\rangle \text{ and } \langle\psi|\Delta\hat{A} = \lambda^*\langle\psi|\Delta\hat{B}$$

b) Can you also show that: $\langle\psi|\left(\Delta\hat{B}\right)^2|\psi\rangle = \frac{1}{2\lambda}\langle\psi|\left[\hat{B},\hat{A}\right]|\psi\rangle$? FYI, this is how we knew that:

$$\lambda = \frac{\langle\psi|[\hat{B},\hat{A}]|\psi\rangle}{2\langle\psi|(\Delta\hat{B})^2|\psi\rangle}$$

c) Using: $\Delta\hat{p}\,|\psi\rangle = \lambda\cdot\Delta\hat{x}|\psi\rangle$, show that the minimum uncertainty wavepacket satisfying: $\langle\psi|[\hat{x},\hat{p}]|\psi\rangle = i\hbar$ is given by:

$$\langle x|\psi\rangle \cong exp\left[\frac{-(x-\langle x\rangle)^2}{4\langle(\Delta\hat{x})^2\rangle} + \frac{i\cdot x\cdot\langle p\rangle}{\hbar}\right]$$

Hint: As above $\lambda = \frac{\langle\psi|[\hat{B},\hat{A}]|\psi\rangle}{2\langle\psi|(\Delta\hat{B})^2|\psi\rangle}$ and you need to use this identity:

$$\frac{-x^2}{4\langle\psi|(\Delta\hat{x})^2|\psi\rangle} + \frac{x\cdot\langle\hat{x}\rangle}{2\langle\psi|(\Delta\hat{x})^2|\psi\rangle} = \frac{-(x-\langle x\rangle)^2}{4\langle(\Delta\hat{x})^2\rangle}$$

Aside: The wavefunction above is not fully correct because it isn't normalized. You don't need to show this, but the normalized wavefunction is:

$$\langle x|\psi\rangle = \left(\frac{1}{2\pi\langle(\Delta\hat{x})^2\rangle}\right)^{\frac{1}{4}} exp\left[\frac{-(x-\langle x\rangle)^2}{4\langle(\Delta\hat{x})^2\rangle} + \frac{i\cdot x\langle p\rangle}{\hbar}\right]$$

Answers: a) A First, we use the fact that $\Delta\hat{A}|\psi\rangle = \lambda\cdot\Delta\hat{B}|\psi\rangle$ below:

$$\langle\psi|(\Delta\hat{A})^2|\psi\rangle = \langle\psi|\Delta\hat{A}\Delta\hat{A}|\psi\rangle = \langle\psi|\Delta\hat{A}\cdot\lambda\cdot\Delta\hat{B}|\psi\rangle$$

Likewise, since $\Delta\hat{B}|\psi\rangle = \frac{1}{\lambda}\Delta\hat{A}|\psi\rangle$: $\langle\psi|(\Delta\hat{B})^2|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{B}|\psi\rangle = \langle\psi|\Delta\hat{B}\cdot\frac{1}{\lambda}\cdot\Delta\hat{A}|\psi\rangle$. Thus:

$$\langle\psi|(\Delta\hat{A})^2|\psi\rangle\langle\psi|(\Delta\hat{B})^2|\psi\rangle = \langle\psi|\Delta\hat{A}\cdot\lambda\cdot\Delta\hat{B}|\psi\rangle\langle\psi|\Delta\hat{B}\cdot\frac{1}{\lambda}\cdot\Delta\hat{A}|\psi\rangle =$$

$$\frac{\lambda}{\lambda}\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle^*\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = |\langle\psi|\Delta\hat{B}\Delta A|\psi\rangle|^2$$

where the fact that $\langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle^*$ is due to the fact that the operators are Hermitian. Next, we have already shown in class that:

$$\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = \frac{1}{2}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle + \frac{1}{2}\langle\psi|\{\Delta\hat{B},\Delta\hat{A}\}|\psi\rangle$$

where the anticommutator is: $\langle\psi|\{\Delta\hat{B},\Delta\hat{A}\}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle + \langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle$, and shown below is why this anticommutator is 0. First, we insert the identities discussed in the problem: $\Delta\hat{B}|\psi\rangle = \frac{1}{\lambda}\Delta\hat{A}|\psi\rangle$ and $\langle\psi|\Delta\hat{A} = \lambda^*\langle\psi|\Delta\hat{B})$, into the above:

$$\langle\psi|\{\Delta\hat{B},\Delta\hat{A}\}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle + \langle\psi|\Delta\hat{A}\Delta\hat{B}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle + \lambda^*\langle\psi|\Delta\hat{B}\cdot\frac{1}{\lambda}\cdot\Delta\hat{A}|\psi\rangle$$

Since $\frac{\lambda^*}{\lambda} = -1$ when λ is imaginary, we are left with:

$$\langle\psi|\{\Delta\hat{B},\Delta\hat{A}\}|\psi\rangle = \langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle - \lambda^*\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = 0$$

Thus, $\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = \frac{1}{2}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle$, which we showed in class is $\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle = \langle\psi|[\hat{B},\hat{A}]|\psi\rangle$ and as a result:

$$\langle\psi|(\Delta\hat{A})^2|\psi\rangle\langle\psi|(\Delta\hat{B})^2|\psi\rangle = \left|\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle\right|^2 = \left|\frac{1}{2}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle\right|^2 = \frac{1}{4}\left|\langle\psi|[\hat{B},\hat{A}]|\psi\rangle\right|^2$$

b) We demonstrated in pt. a that:

$$\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = \frac{1}{2}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle$$

Since $\langle\psi|\Delta\hat{B}\Delta\hat{A}|\psi\rangle = \lambda\langle\psi|\Delta\hat{B}\Delta\hat{B}|\psi\rangle = \frac{1}{2}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle$, we are left with:

$$\langle\psi|\Delta\hat{B}\Delta\hat{B}|\psi\rangle = \langle\psi|(\Delta\hat{B})^2|\psi\rangle = \frac{1}{2\lambda}\langle\psi|[\Delta\hat{B},\Delta\hat{A}]|\psi\rangle$$

c) First start with $\Delta\hat{p}\,|\psi\rangle = \lambda\cdot\Delta\hat{x}|\psi\rangle$ and apply a $\langle x|$ bra to the left:

$$\langle x|\Delta\hat{p}\,|\psi\rangle = \langle x|\lambda\cdot\Delta\hat{x}|\psi\rangle = \left(-i\hbar\frac{\partial}{\partial x} - \langle\hat{p}\rangle\right)\langle x|\psi\rangle = \lambda(\hat{x} - \langle\hat{x}\rangle)\langle x|\psi\rangle$$

As a result:$-i\hbar\frac{\partial}{\partial x}\psi(x) - \langle\hat{p}\rangle\psi(x) = \lambda\cdot x\cdot\psi(x) - \langle\hat{x}\rangle\psi(x)$. We can factor this as:

$$\frac{\partial\psi(x)}{\partial x} = \psi'(x) = \left(\frac{i\lambda}{\hbar}\cdot x + \langle\hat{p}\rangle - \langle\hat{x}\rangle\right)\psi(x)$$

We can divide out by to get the log derivative on the left:

$$\int\frac{\psi'(x)}{\psi(x)} = \ln(\psi(x)) = \int\frac{i}{\hbar}(\lambda\cdot x + \langle\hat{p}\rangle - \lambda\cdot\langle\hat{x}\rangle)\partial x$$

Thus, $\psi(x) = exp\int\frac{i}{\hbar}(\lambda\cdot x + \langle\hat{p}\rangle - \lambda\cdot\langle\hat{x}\rangle)\partial x = \exp\left(\frac{i}{\hbar}\left(\frac{\lambda}{2}\cdot x^2 + x\cdot\langle\hat{p}\rangle - \lambda\cdot x\cdot\langle\hat{x}\rangle\right)\right)$. Here we can use the identity
$\lambda = \frac{\langle\psi|[\hat{B},\hat{A}]|\psi\rangle}{2\langle\psi|\Delta\hat{B}^2|\psi\rangle} = \frac{\langle\psi|[\hat{x},\hat{p}]|\psi\rangle}{2\langle\psi|\Delta\hat{x}^2|\psi\rangle} = \frac{i\hbar}{2\langle\psi|\Delta\hat{x}^2|\psi\rangle}$ and as a result:

$$\psi(x) = \exp\left(\frac{i}{\hbar}\left(\frac{i\hbar \cdot x^2}{4\langle\psi|\Delta\hat{x}^2|\psi\rangle} + x \cdot \langle\hat{p}\rangle - \frac{i\hbar \cdot x \cdot \langle\hat{x}\rangle}{2\langle\psi|\Delta\hat{x}^2|\psi\rangle}\right)\right)$$
$$= \exp\left(\frac{-x^2}{4\langle\psi|\Delta\hat{x}^2|\psi\rangle} + \frac{i}{\hbar} \cdot x \cdot \langle\hat{p}\rangle + \frac{x \cdot \langle\hat{x}\rangle}{2\langle\psi|\Delta\hat{x}^2|\psi\rangle}\right)$$

Now we can use the identity $\frac{-x^2}{4\langle\psi|\Delta\hat{x}^2|\psi\rangle} + \frac{x \cdot \langle\hat{x}\rangle}{2\langle\psi|\Delta\hat{x}^2|\psi\rangle} = \frac{-(x-\langle x\rangle)^2}{4\langle\Delta\hat{x}^2\rangle}$:

$$\psi(x) = \exp\left(\frac{-(x-\langle x\rangle)^2}{4\langle\Delta\hat{x}^2\rangle} + \frac{i}{\hbar} \cdot x \cdot \langle\hat{p}\rangle\right)$$

If you normalize it you would find that the normalization constant is $\left(\frac{1}{2\pi\langle\Delta\hat{x}^2\rangle}\right)^{\frac{1}{4}}$.

4. Consider a particle of mass m in a linear potential of the form $V(x) = E - F(x-a)$, where E is energy while F and a are constants. By solving the Schrodinger Equation in the momentum representation:

$$\left\{\frac{p^2}{2m} + E - F(x-a)\right\}\psi(p) = E\psi(p)$$

show that the wavefunction $\psi(x)$ is given by:

$$\psi(x) = C\int_{-\infty}^{\infty} exp\left[\frac{1}{i\hbar}\left(\frac{p^3}{6mF} + p(a-x)\right)\right]\partial p$$

Hint: I was Googling how to solve the above, and I ran into this website which states:

For the 1st order homogeneous differential equation:

$$\frac{\partial y}{\partial t} + p(t)y = 0$$

The solution is: $\int \frac{\partial y}{y} = \int -p(t)\partial t$.

Now all you have to do is “translate” the p’s, y’s and t’s to conform to your problem.

Answer: The idea is to use the fact that $x = i\hbar\frac{\partial}{\partial p}$ and then use the general form for a 1st order differential equation:

$$\frac{p^2}{2m}\psi(p) + E\psi(p) - i\hbar F\frac{\partial}{\partial p}\psi(p) - Fa\psi(p) = E\psi(p)$$

Note how the cancel leaving:

$$i\hbar F\frac{\partial\psi}{\partial p}-\left(\frac{p^2}{2m}+Fa\right)\psi(p)=0$$

Therefore:

$$\frac{\partial\psi(p)}{\psi(p)}=\frac{1}{i\hbar}\left(\frac{p^2}{2mF}+a\right)\partial p$$

Now integrate both sides: $\int\frac{\partial\psi(p)}{\psi(p)}=\ln\left(\psi(p)\right)$ while the other side is: $\int\frac{1}{i\hbar}\left(\frac{p^2}{2miF}+a\right)\partial p=\frac{1}{i\hbar}\left(\frac{p^3}{6mF}+pa\right)$ and thus $\psi(p)=\exp\left\{\frac{1}{i\hbar}\left(\frac{p^3}{6mF}+pa\right)\right\}$. Next, you transform the momentum representation into space via:

$$\psi(x)=\int_{-\infty}^{\infty}\exp\left(\frac{ipx}{\hbar}\right)\psi(p)\partial p=\int_{-\infty}^{\infty}\exp\left(i\frac{px}{\hbar}\right)\exp\left\{\frac{1}{i\hbar}\left(\frac{p^3}{6mF}+pa\right)\right\}\partial p=$$

$$\int_{-\infty}^{\infty}\exp\left\{\frac{1}{i\hbar}\left(\frac{p^3}{6mF}+pa\right)-\frac{px}{i\hbar}\right\}\partial p=\int_{-\infty}^{\infty}\exp\left\{\frac{1}{i\hbar}\left(\frac{p^3}{6mF}+p(a-x)\right)\right\}\partial p$$

5. Extra Maths Learnings! Let $\hat{A}$ and $\hat{B}$ be any operators and λ is a constant. Please show that:

$$\hat{f}(\lambda)=e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}}=\hat{A}+\lambda\left[\hat{A},\hat{B}\right]+\frac{1}{2!}\lambda^2\left[\left[\hat{A},\hat{B}\right],\hat{B}\right]+\cdots=\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n$$

where $\left[\hat{A},\hat{B}\right]_n$ is the n-fold commutator of $\hat{A}$ and $\hat{B}$, i.e. $\left[\hat{A},\hat{B}\right]_n=\left[\left[\hat{A},\hat{B}\right]_{n-1},\hat{B}\right]$

Hint: It is tempting to expand the exponentials into their Taylor series, FOIL them out, and show that the expression is consistent with $\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n$. However, we will go a different way, which is to show consistency with the uniqueness theorem from last problem set.

a) First, take the derivative of $f(\lambda)=e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}}$ and show that the result is consistent with:

$$f'(\lambda)=\left[\hat{f},\hat{B}\right]$$

2nd Hint: You can rearrange an operator: $\hat{B}e^{\lambda\hat{B}}$ as: $e^{\lambda\hat{B}}\hat{B}$ since an operator and function of that same operator commute.

b) Next, show that the derivative of $\hat{f}(\lambda) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n$ is consistent with the result from pt. a. I hope you remember that $\frac{n}{n!} = \frac{1}{(n-1)!}$, and here is a useful identity:

$$\left[\hat{f},\hat{B}\right] = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_{n+1}$$

Aside: The uniqueness theorem states that there can only be one solution to a differential equation under certain conditions. Here, you are showing that the differential equations $\frac{\partial}{\partial\lambda}e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}}$ and $\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n$ are both $f'(\lambda) = \left[\hat{f},\hat{B}\right]$, and thus it must be true that the solutions to $f'(\lambda)$: $e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}}$ and $\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n$, are in fact the same.

Answer: a) The derivative is:

$$f'(\lambda) = -\hat{B}e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}} + e^{-\lambda\hat{B}}\hat{A}\hat{B}e^{\lambda\hat{B}} = -\hat{B}e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}} + e^{-\lambda\hat{B}}\hat{A}e^{\lambda\hat{B}}\hat{B}$$

Given that I gave you the answer, you can easily show that:

$$-\hat{B}\hat{f}(\lambda) + \hat{f}(\lambda)\hat{B} = \left[\hat{f},\hat{B}\right]$$

b) Next, take the other derivative:

$$\frac{\partial}{\partial\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\left[\hat{A},\hat{B}\right]_n = \sum_{n=0}^{\infty}n\frac{\lambda^{n-1}}{n!}\left[\hat{A},\hat{B}\right]_n = \sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}\left[\hat{A},\hat{B}\right]_n$$

If that last step confused you, note that in the middle expression that the n=0 is 0, so you don't need to include it in the summation. Also we used the fact that: $\frac{n}{n!} = \frac{1}{(n-1)!}$. Last, we can use the fact that we can switch n'=n-1 as:

$$\sum_{n=1}^{\infty}\frac{\lambda^{n-1}}{(n-1)!}\left[\hat{A},\hat{B}\right]_n = \sum_{n'=0}^{\infty}\frac{\lambda^{n'}}{n'!}\left[\hat{A},\hat{B}\right]_{n'+1} = \left[\hat{f},\hat{B}\right] = f'(\lambda)$$

so the proof is complete.